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Einstein–Maxwell Null Fields with Non-Zero Twist

P. A. GOODINSON

Department of Mathematics, Queen Elizabeth College, London

and R. A. NEWING

Department of Applied Mathematics, University College of North Wales, Bangor

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Abstract

It is shown that if l^a is the propagation vector of a null Maxwell field in a space-time with metric $\mathring{g}_{ab}$, then it is also the propagation vector in the space-time $g_{ab} = \mathring{g}_{ab} + 2f\mathring{l}_a\mathring{l}_b$. This result, together with the Robinson metric for vacuum gravitational fields and Hughston's generalization to radiating fields, is used to set up equations for combined gravitational and electromagnetic null-fields with special reference to fields with non-zero twist.

1. *Introduction*

A Maxwell null field with zero current implies the existence of a shear-free geodesic congruence. This is also true for an Einstein–Maxwell field with which is associated a non-zero null current distribution (Goodinson & Newing, 1970). When the congruence is also twist free, the metric of space-time may be reduced to a standard form (Wyman & Trollope, 1965; Geroch, 1966; Goodinson & Newing, 1970). When the congruence is diverging with non-zero twist, a standard form for the space-time metric is provided by the Robinson metric (Robinson *et al.*, 1969; Robinson & Robinson, 1969). Hughston (1971) has used the Robinson metric to construct a generalized Vaidya radiating metric and this present paper will show that such metrics include metrics admitting Einstein–Maxwell null fields with non-zero twist.

In this paper, the notation and definitions of the Robinson and Hughston papers are adopted.

2. *Maxwell Null Fields in Related Space-Times*

Consider space-times defined by the metric tensors $\mathring{g}_{ab}$ and $g_{ab} = \mathring{g}_{ab} + 2f\mathring{l}_a\mathring{l}_b$, where $\mathring{l}_a$ is a null vector in the first space-time, then with signature +2,

$$\mathring{g}_{ab} = 2\mathring{m}_{[a}\mathring{\bar{m}}_{b]} - 2\mathring{l}_{[a}\mathring{n}_{b]}$$

where $\mathring{l}_a, \mathring{n}_a, \mathring{m}_a, \mathring{\bar{m}}_a$ is a null tetrad such that for real coordinates, $\mathring{l}_a \mathring{n}^a = -1$, $\mathring{m}_a \mathring{\bar{m}}^a = 1$.

The corresponding tetrad for g_{ab} may be taken to be

$$l_a = \mathring{l}_a, \qquad m_a = \mathring{m}_a, \qquad n_a = \mathring{n}_a - f\mathring{l}_a$$

the contravariant vectors being

$$l^a = \mathring{l}^a, \qquad m^a = \mathring{m}^a, \qquad n^a = \mathring{n}^a + f\mathring{l}^a$$

Suppose $\mathring{k}^a$ is parallel to the propagation vector of a Maxwell null field in space-time with metric $\mathring{g}_{ab}$,then choosing $\mathring{m}_a$ so that $\mathring{l}_{[a}\mathring{m}_{b]}$ is self-dual, the Maxwell equations are

$$\frac{2}{\sqrt{\mathring{g}}}(\sqrt{(\mathring{g})}\,\lambda\, e^{i\phi}\,\mathring{l}^{[a}\mathring{m}^{b]})_{,b} = j^a \tag{2.1}$$

where j^a is a real vector in the case of a real coordinate system.

Since the determinant $\mathring{g}$ of the metric tensor $\mathring{g}_{ab}$ is equal to that of g_{ab}, (2.1) are also the Maxwell equations for the space-time with metric tensor g_{ab}.

Theorem

If the null vectors $\mathring{l}_a$, $\mathring{m}_a$ define a Maxwell null field in space-time with metric tensor $\mathring{g}_{ab}$, then these vectors also define a Maxwell field in space-time with metric $\mathring{g}_{ab} + 2f\mathring{l}_a\mathring{l}_b$.

Consider now the Robinson metric

$$ds^2 = 2P^2\, d\xi d\tilde{\xi} + 2(d\rho + Zd\xi + \tilde{Z}d\tilde{\xi})\, d\Sigma + 2S(d\Sigma)^2 \tag{2.2}$$
$$d\Sigma = a(bd\xi + \tilde{b}d\tilde{\xi} + d\sigma) = k_a\, dx^a$$

where ρ and σ are real coordinates and ξ is complex. In this coordinate system, null tetrad vectors $\mathring{l}_a, \mathring{n}_a, \mathring{m}_a, \mathring{\bar{m}}_a$ for the associated metric $d\mathring{s}^2$ defined by $S = 0$ may be taken to be respectively

$$\mathring{k}_a = a(b\delta_a{}^1 + \tilde{b}\delta_a{}^2 + a\delta_a{}^3)$$
$$\mathring{s}_a = -Z\delta_a{}^1 - \tilde{Z}\delta_a{}^2 - \delta_a{}^4$$
$$\mathring{t}_a = P\delta_a{}^1, \qquad \mathring{v}_a = \delta_a{}^2$$

the coordinates ξ, $\tilde{\xi}$, σ, ρ being labelled from 1 to 4.

The Maxwell equations for a null field with propagation vector parallel to k_a in space-time (2.2) are therefore

$$\begin{aligned}\sqrt{(\mathring{g})}j^a &= 2(\sqrt{(\mathring{g})}\,\lambda\, e^{i\phi}\, k^{[a}t^{b]})_{,b}\\ &= -\delta_2{}^a(X e^{i\phi})_{,4} + \delta_3{}^a(\tilde{b}X e^{i\phi})_{,4}\\ &\qquad + \delta_4{}^a[(X e^{i\phi})_{,2} - (\tilde{b}X e^{i\phi})_{,3}]\end{aligned}$$

where $X = aP\lambda$ and $\mathring{g} = a^2P^4$.

Now $j_a dx^a$ is invariant with respect to coordinate transformations and so $j_a dx^a$ must be real. This reality condition implies that the contravariant components of $\mathring{j}^a$ are such that $\mathring{j}^1 = \mathring{j}^2$ with $\mathring{j}^3$ and $\mathring{j}^4$ both real. In the space-time g_{ab} these results become $\tilde{j}^1 = j^2, j^3, j^4$ real which gives

$$(X e^{i\phi})_{,4} = 0 \tag{2.3}$$

together with $b_{,4} = 0$.

The Maxwell equations for the null field then become

$$\sqrt{(\mathring{g})} j^a = \delta_4{}^a[(X e^{i\phi})_{,2} - (\tilde{b} X e^{i\phi})_{,3}] \tag{2.4}$$

with $\sqrt{(\mathring{g})} j^a = \sqrt{(\mathring{g})} \mathring{j}^a$.

3. *Generalized Vaidya Metrics*

Robinson *et al.* have shown that the general metric admitting a null geodesic shear-free vector field may be expressed in the form (2.2) with P, Z, $\tilde{Z}$, S functions of all four coordinates and a, b, $\tilde{b}$ independent of ρ, the dilation and twist being given by

$$\theta = (\ln P)_{,4}, \qquad \omega = -\frac{ia}{2} P^{-2}\, Y \qquad \text{where } Y = i(\tilde{b} \; - b_2)$$

Hughston has shown that if (2.2) defines a vacuum gravitational field, then the metric

$$d\check{s}^2 = ds^2 + 2H\rho^4(\rho^2 + \Omega^2)^{-1}(k_a\, dx^a)^2 \tag{3.1}$$

defines a radiation field, the Ricci tensor being

$$\check{R}_{ab} = (\rho^2 + \Omega^2)^{-1} \rho^4 H_{/3}\, k_a k_b \qquad \text{where } H_{/1} = 0,\ H_{/2} = 0,\ H_{/4} = -3H$$

These conditions imply that

$$G_{,3} = 0 \tag{3.2}$$

$$\text{where } G := (\ln Y)_1 + b_{,3}$$

this condition constituting a restriction on the parameter b. The parameters a, b, u are arbitrary functions of ξ, $\tilde{\xi}$, σ as far as the main equations are concerned. These functions are related by the subsidiary equations and for generalized Vaidya fields, b is further restricted by (3.2).

For the case of non-zero twist, $Y \neq 0$ and Hughston has obtained the solution

$$H = h(\rho a Y)^{-3}$$

where $h = \eta \exp\{3 \int (G d\xi + \tilde{G} d\tilde{\xi})\}$ and η is a constant. Now

$$H_{/3} = \rho^{-1}(H_3 - u_3 H_{/4}) = -3H[\ln(a\rho^{-1}\, Y)]_{/3}$$

$$= \frac{-3h}{a^4 \rho^4\, Y^3}[\ln(a e^{-u}\, Y)]_{,3}$$

and hence

$$\check{R}_{ab} = \frac{-3h}{(\rho^2 + \Omega^2)\, a^4\, Y^3}[\ln(a e^{-u}\, Y)]_{,3}\, k_a k_b$$

In suitable units, the energy-momentum tensor E_{ab} for the Maxwell field of Section 2 is $\lambda^2 k_a k_b$, and the gravitational equations $\check{R}_{ab} + E_{ab} = 0$ are satisfied if

$$\lambda^2 = \frac{3h}{(\rho^2 + \Omega^2) a^4 Y^3} [\ln(a\,e^{-u}\, Y)]_{,3}$$

so that

$$X^2 = a^2 P^2 \lambda^2 = \frac{-3h}{2Y} [a\,e^{-u}\, Y)^{-2}]_{,3} \tag{3.3}$$

The conditions for the existence of an Einstein–Maxwell field are then given by (2.4) and (3.3), with b subject to the restriction (3.2).

4. *The Auxiliary Conditions*

With reference to Robinson and Robinson, auxiliary conditions which are required to be satisfied are given in terms of defined quantities A and B:

$$A = (m - iM)_1 + 3\Lambda(m - iM)$$
$$B = \exp(-3u)[\exp(3u)(m + iM)]_3 + \exp(-4u).\ I$$

where $I = J_{22} + 2\tilde{L}J_2$, $J = L_1 + L^2$.

Further, Robinson and Robinson obtain expressions for M and J in terms of a potential U where U is such that $U_{,3} = a\,e^{-u}$

$$\left.\begin{aligned} M &= \frac{i}{2}\exp(-3u)(U_{1122} - U_{2211}) \\ J &= e^{u}\, U_{113} \end{aligned}\right\} \tag{4.1}$$

Particular Solution

Consider now equation (3.3). In order to obtain a particular class of solutions, the restriction (3.2) may be satisfied by taking $b_{,3} = 0$ and in this case $h = \eta Y^3$. Making use of the potential U, we have

$$X^2 = 3\eta (U_{,3})^{-3} U_{,33}$$

Defining a real coordinate x to be $x = \xi + \bar{\xi}$, then particular solutions for b and U may be chosen to be

$$\left.\begin{aligned} b &= i(1 + x^2)^{-1} \\ U &= e^{\sigma}\sqrt{(1 + x^2)} \end{aligned}\right\} \tag{4.2}$$

This choice of U is such that

$$U_{11} = 0, \qquad U_{113} = 0, \qquad U_{1122} = 0$$

and it is therefore obvious from (4.1) that both M and J are zero. A and B can thus be made to vanish if m is put equal to zero, and so the auxiliary conditions $A=0$, $B=0$ corresponding to a vacuum background metric, can be satisfied.

5. *The Maxwell Equations*

From Section 2, the current is given by

$$\surd(\mathring{g})j^a = \delta_4{}^a[(Xe^{i\phi})_{,2} - (\bar{b}Xe^{i\phi})_{,3}]$$

if the complexion ϕ is such that $\phi = \phi(x)$. With the choice (4.2) for b and U, it follows that

$$X^2 = 3\eta\, e^{-2\sigma}(1+x^2)^{-1}$$

$$X_{,2} = -Xx(1+x^2)^{-1}, \qquad X_{,3} = -X, \qquad Y = -4x(1+x^2)^{-2} \qquad (\neq 0)$$

and so

$$\surd(\mathring{g})j^a = \delta_4{}^a\, Xe^{i\phi}(1+x^2)^{-1}[-x + i\{(1+x^2)\,\phi' - 1\}]$$

where $\phi' = d\phi/dx$. The reality condition for j^a implies that

$$(1+x^2)\,\phi' - 1 = x\tan\phi \qquad (5.1)$$

thus giving a real current

$$\surd(\mathring{g})j^a = X(1+x^2)^{-1}[x^2 + \{(1+x^2)\,\phi' - 1\}^2]^{1/2}\,\delta_4{}^a \qquad (5.2)$$

Defining quantities p and q such that

$$p = \cos\theta\sin\phi \qquad q = \cos\theta\cos\phi$$

and putting $x = \tan\theta$, equation (5.1) can be re-written as

$$\frac{dp}{d\theta} = q$$

and so p must satisfy

$$\left(\frac{dp}{d\theta}\right)^2 + p^2 = \cos^2\theta$$

The complexion ϕ can now be expressed in terms of x and p by the equation

$$\phi = \sin^{-1}\{p(1+x^2)^{1/2}\} \qquad (5.3)$$

and (5.2) can be written

$$\surd(\mathring{g})j^a = x\,e^{-\sigma}(1-x^2)^{-3/2}\sec\phi\,\delta_4{}^a \qquad (5.4)$$

i.e. j^a in terms of x, σ, p.

Thus a particular solution of (2.4), (3.2), (3.3) is provided by equations (4.2) and (5.3) where p is a solution of the equation

$$\left(\frac{dp}{d\theta}\right)^2 + p^2 = \cos^2\theta$$

and so an Einstein–Maxwell null field exists and the current density vector is given by (5.4).

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